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# On some transformations between positive self–similar Markov processes

**Loïc CHAUMONT**[*] **Víctor RIVERO**[†]

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**Abstract**

A path decomposition at the infimum for positive self-similar Markov processes (pssMp) is obtained. Next, several aspects of the conditioning to hit 0 of a pssMp are studied. Associated to a given a pssMp $X$, that never hits 0, we construct a pssMp $X^{\downarrow}$ that hits 0 in a finite time. The latter can be viewed as $X$ conditioned to hit 0 in a finite time and we prove that this conditioning is determined by the pre-minimum part of $X$. Finally, we provide a method for conditioning a pssMp that hits 0 by a jump to do it continuously.

*Key words*: Self-similar Markov processes, Lévy processes, weak convergence, decomposition at the minimum, conditioning, $h$-transforms.

**MSC:** 60 G 18 (60 G 17).

## 1 Introduction

This work concerns positive self–similar Markov processes (pssMp), that is $[0,\infty[$-valued strong Markov processes that have the scaling property: there exists an $\alpha > 0$ such that for any $0 < c < \infty$,

$$\{(cX_{tc^{-1/\alpha}}, t \geq 0), \mathbb{P}_x\} \stackrel{(d)}{=} \{(X_t, t \geq 0), \mathbb{P}_{cx}\}, \qquad x > 0.$$

This class of processes has been introduced by Lamperti [22] and since then studied by several authors, see e.g. [4, 6, 7, 10, 11, 25, 26]. We will make systematic use of a result due to Lamperti that establishes that any pssMp is the exponential of a Lévy process time changed, this will be recalled at Section 2.

Some of the motivations of this work are some path decompositions and conditionings that can be deduced from [9, 10] and that we will recall below, in the particular case where the positive self–similar Markov process is a stable Lévy process conditioned to stay positive.

[*]Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4, Place Jussieu - 75252 Paris Cedex 05, France.

[†]CIMAT A.C. Calle Valenciana s/n, C.P.36240, Guanajuato, Gto. Mexico; and Université Paris X, Nanterre, MODAL'X, 200, Avenue de la République, 92001, Nanterre Cedex, France; Email: rivero@cimat.mx

Let $\widetilde{\mathbb{P}}$ be a law on the the space of càdlàg paths under which the canonical process $X$, is an $\alpha$-stable Lévy process, $0 < \alpha \leq 2$, i.e. a process with independent and stationary increments that is $1/\alpha$-self–similar. Associated to this process we can construct a pssMp, say $(X, \mathbb{P})$, that can be viewed as $(X, \widetilde{\mathbb{P}})$ conditioned to stay positive. The construction can be performed either via the Tanaka–Doney [15] path transform of $(X, \widetilde{\mathbb{P}})$ or as an $h$-transform of the law of $(X, \widetilde{\mathbb{P}})$ as in [9, 13] or also, in the spectrally one sided case, via Bertoin's transformation [2].

Another interesting process related to $(X, \widetilde{\mathbb{P}})$ is $(X, \mathbb{P}^{\downarrow}_{\cdot})$, which was introduced in [9], can be viewed as $(X, \widetilde{\mathbb{P}})$ conditioned to hit 0 continuously and is constructed via an h-transform of $(X, \widetilde{\mathbb{P}})$ killed at its first hitting time of $]-\infty, 0]$.

Using the results of Millar [24], in [9] it has been proved the following results for $(X, \mathbb{P})$, relating $(X, \mathbb{P}^{\downarrow})$ and $(X, \mathbb{P})$ started at 0, with the pre and post minimum parts of $(X, \mathbb{P})$.

**Fact 1.** *Let $I^X = \inf\{X_s, s > 0\}$ and $m = \sup\{t > 0 : X_{t-} \wedge X_t = I^X\}$. Under $\mathbb{P}$, the pre-minimum part of $X$, i.e. $\{X_t, 0 \leq t < m\}$, and the post minimum part of $X$, i.e. $\{X_{m+t}, t > 0\}$ are conditionally independent given the value of $I^X$. For any $x > 0$, under $\mathbb{P}_x$, conditionally on $I^X = y$, $0 < y \leq x$, the law of the former is $(X + y,\, \mathbb{P}^{\downarrow}_{x-y})$ and that of the later is $(X + y,\, \mathbb{P}_{0+})$, where $\mathbb{P}_{0+}$ is the limiting law of $(X, \mathbb{P}_{\cdot})$ as the starting point tends to 0, $\mathbb{P}_x \xrightarrow{w} \mathbb{P}_{0+}$ as $x \to 0+$.*

Furthermore, it can be verified using the previous result, and it is intuitively clear, that under $\mathbb{P}_x$ the law of the pre-minimum (respectively, post-minimum) of $X$, conditionally on the event $\{I^X < \epsilon\}$, converges as $\epsilon \to 0$, to the law $\mathbb{P}^{\downarrow}_x$, respectively $\mathbb{P}_{0+}$, in the sense that,

**Fact 2.** $\lim_{\epsilon \to 0+} \mathbb{P}_x(F \cap \{t < m\}, G \circ \theta_m | I^X < \epsilon) = \mathbb{P}^{\downarrow}_x(F \cap \{t < T_0\})\, \mathbb{P}_{0+}(G), \quad F \in \mathcal{G}_t, G \in \mathcal{G}_\infty,$ *where $\{\mathcal{G}_t, t \geq 0\}$ is the natural filtration generated by $X$.*

Our first purpose is to extend Facts 1 & 2, to a larger class of positive self-similar Markov processes. That is the content of sections 3 & 4, respectively.

Here is another interpretation of the law $\mathbb{P}^{\downarrow}$. Let $\mathbb{P}^0$ be the law of the process $(X, \widetilde{\mathbb{P}})$ killed at its first hitting time of $]-\infty, 0]$. The process $(X, \mathbb{P}^0)$ still has the strong Markov property and inherits the scaling property from $(X, \widetilde{\mathbb{P}})$, so it is a pssMp and it hits 0 in a finite time. Moreover, whenever $(X, \widetilde{\mathbb{P}})$ has negative jumps, the process $(Y, \mathbb{P}^0)$ hits 0 for the first time with a negative jump:

$$\mathbb{P}^0_x(T_0 < \infty, X_{T_0-} > 0) = 1, \qquad \forall x > 0,$$

where $T_0 = \inf\{t > 0 : X_t = 0\}$. It has been proved in [9], that $\mathbb{P}^{\downarrow}$ is an $h$ transform of $\mathbb{P}^0$ via the excessive function $x \mapsto x^{\alpha(1-\rho)-1}$, $x > 0$, where $\rho$ is the positivity parameter of $(X, \widetilde{\mathbb{P}})$, $\rho = \widetilde{\mathbb{P}}(X_1 \geq 0)$. Furthermore, $(X, \mathbb{P}^{\downarrow})$ hits 0 continuously and in a finite time, i.e.:

$$\mathbb{P}^{\downarrow}_x(T_0 < \infty, X_{T_0-} = 0) = 1, \qquad \forall x > 0,$$

and Proposition 3 in [9] describes a relationship between $\mathbb{P}^{\downarrow}$ and $\mathbb{P}^0$ that allows us to refer to $\mathbb{P}^{\downarrow}$ as the law of $(X, \mathbb{P}^0)$ conditioned to hit 0 continuously. The latter conditioning is performed by approximating the set $\{I^{Y^0} = 0\}$ by the sequence of sets $\{I^{Y^0} < \epsilon\}, \epsilon > 0$.

In Section 5, we obtain an analogous result for a larger class of self–similar Markov processes. Namely those associated to a Lévy process killed at an independent exponential time and which satisfy a Cramér's type condition. Furthermore, an alternative method for conditioning a self-similar Markov process that hits 0 by a jump, to hit 0 continuously, is provided by making tend to 0 the height of the jump by which the process hits the state 0.

The approach used to aboard these problems is based on Lamperti's representation between real valued Lévy processes and pssMp which we recall in the following section.

# 2 Some preliminaries on pssMp

Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0,\infty)$, with values in $\mathbb{R} \cup \Delta$, where $\Delta$ is a cemetery point. Each path $\omega \in \mathbb{D}$ is such that $\omega_t = \Delta$, for any $t \geq \inf\{t : \omega_t = \Delta\} := \zeta(\omega)$. As usual we extend the functions $f : \mathbb{R} \to \mathbb{R}$ to $\mathbb{R} \cup \Delta$ by $f(\Delta) = 0$. The space $\mathbb{D}$ is endowed with the Skohorod topology and its Borel $\sigma$-field. We will denote by $X$ the canonical process of the coordinates and $(\mathcal{F}_t)$ will be the natural filtration generated by $X$. Moreover, let $\mathbf{P}$ be a reference probability measure on $\mathbb{D}$ under which the process, $\xi$, is a Lévy process; we will denote by $(\mathcal{D}_t, t \geq 0)$, the complete filtration generated by $\xi$.

Fix $\alpha > 0$ and let $(\mathbb{P}_x, x > 0)$ be the laws of an $\alpha$-pssMp associated to $(\xi, \mathbf{P})$ via the Lamperti representation. Formally, define

$$A_t = \int_0^t \exp\{(1/\alpha)\xi_s\} ds, \qquad t \geq 0,$$

and let $\tau(t)$ be its inverse,

$$\tau(t) = \inf\{s > 0 : A_s > t\},$$

with the usual convention, $\inf\{\emptyset\} = \infty$. For $x > 0$, we denote by $\mathbb{P}_x$ the law of the process

$$x \exp\{\xi_{\tau(tx^{-1/\alpha})}\}, \qquad t > 0,$$

with the convention that the above quantity is $\Delta$ if $\tau(tx^{-1/\alpha}) = \infty$. The Lamperti representation ensures that the laws $(\mathbb{P}_x, x > 0)$ are those of a pssMp with index of self-similarity $\alpha$.

Besides, recall that any Lévy process $(\xi, \mathbf{P})$ with lifetime has the same law as a Lévy process with infinite lifetime that has been killed at a rate $\mathrm{q} \geq 0$. It follows that $T_0 = \inf\{t > 0 : X_t = 0\}$ has the same law under $\mathbb{P}_x$ as $x^{1/\alpha} A_\zeta$ under $\mathbf{P}$ with

$$A_\zeta = \int_0^\zeta \exp\{(1/\alpha)\xi_s\} ds.$$

So, if $\mathrm{q} > 0$, then the random variable $A_\zeta$ is a.s. finite; while in the case $\mathrm{q} = 0$, we have two possibilities, either $A_\zeta$ is finite a.s. or infinite a.s.; the former happens if and only if $\lim_{t\to\infty} \xi_t = -\infty$, a.s. and the latter if and only if $\limsup_{t\to\infty} \xi_t = \infty$, a.s.

Lamperti proved that any pssMp can be constructed this way and obtained the following classification of pssMp's:

(LC1) $\mathrm{q} > 0$, if and only if

$$\mathbb{P}_x(T_0 < \infty,\ X_{T_0-} > 0,\ X_{T_0+s} = 0,\ \forall s \geq 0) = 1, \qquad \text{for all } x > 0. \tag{2.1}$$

(LC2) q $= 0$ and $\lim_{t\to\infty}\xi_t = -\infty$ a.s. if and only if

$$\mathbb{P}_x(T_0 < \infty,\ X_{T_0-} = 0,\ X_{T_0+s} = 0,\ \forall s \geq 0) = 1, \qquad \text{for all } x > 0, \tag{2.2}$$

(LC3) q $= 0$ and $\limsup_{t\to\infty}\xi_t = \infty$ a.s. if and only if

$$\mathbb{P}_x(T_0 = \infty) = 1, \qquad \text{for all } x > 0. \tag{2.3}$$

Observe that without loss of generality we can and we will suppose that $\alpha = 1$ in Lamperti's construction of pssMp because all our results may trivially be extended to any $\alpha > 0$ by considering $X^\alpha$ which is a pssMp with index of self-similarity $\alpha$.

In this work we will be mostly interested by those pssMp that belong to the class LC3; nevertheless, in Section 5 we will prove that some elements of the class LC1 can be transformed into elements of the class LC2.

# 3 Path decomposition at the minimum

We suppose throughout this section that $(\xi, \mathbf{P})$ is a Lévy process with infinite lifetime which drifts to $+\infty$, that is $\lim_{t\to+\infty}\xi_t = +\infty$, a.s. We start by recalling a William's type path decomposition of $(\xi, \mathbf{P})$ at its minimum. Let $I^\xi = \inf_{t\geq 0}\xi_t$ and $\rho = \sup\{t : \xi_t \wedge \xi_{t-} = I^\xi\}$. We define the post minimum process as

$$\underset{\rightarrow}{\xi} \overset{\text{(def)}}{=} (\xi_{\rho+t} - I^\xi,\ t \geq 0)\,.$$

The following result is due to Millar [24], proposition 3.1 and Theorem 3.2.

**Theorem 1.** *The pre-minimum process* $((\xi_t,\ t < m), \mathbf{P})$ *and the post-minimum process* $(\underset{\rightarrow}{\xi}, \mathbf{P})$ *are independent. Moreover, the three following exhaustive cases hold:*

*(i)* $0$ *is regular for both* $(-\infty, 0)$ *and* $(0, \infty)$ *and* $\mathbf{P}$*-a.s., there is no jump at the minimum,*

*(ii)* $0$ *is regular for* $(-\infty, 0)$ *but not for* $(0, \infty)$ *and* $I^\xi = \xi_{\rho-} < \xi_\rho$ , $\mathbf{P}$*-a.s.*

*(iii)* $0$ *is regular for* $(0, \infty)$ *but not for* $(-\infty, 0)$ *and* $I^\xi = \xi_\rho < \xi_{\rho-}$, $\mathbf{P}$*-a.s.*

*In any case under* $\mathbf{P}$*, the process* $(\xi_t,\ t < m)$ *and* $\underset{\rightarrow}{\xi}$ *are also conditionally independent given* $I^\xi$ *and the process* $\underset{\rightarrow}{\xi}$ *is strongly Markovian.*

Actually, Millar's result is much more general and asserts that for any Markov process, which admits a minimum, the pre-minimum process and the post minimum process are conditionally independent given both the value at the minimum and the subsequent jump and the post-minimum process is strongly Markovian. When this Markov process is a pssMp that belongs to the class (LC3), we may complete Millar's result as in the following proposition. First of all, observe that $X$ derives towards $+\infty$ as well as $\xi$, and so the following are well defined $I^X = \inf_{t\geq 0} X_t$ and $m = \sup\{t : X_t \wedge X_{t-} = I^X\}$.

**Proposition 1.** *For any $x>0$, under $\mathbb{P}_x$, the processes $(X_t, t<m)$ and $(X_{t+m},\, t\ge 0)$ are conditionally independent given $I^X$, and with the representation given by Lamperti's transformation (Section 2), we have*

$$((X_t, 0\le t<m),\mathbb{P}_x) \;=\; \left(\left(x\exp\xi_{\tau(t/x)},\, 0\le t< x\int_0^\rho \exp\xi_s \mathrm{d}s\right),\mathbf{P}\right), \tag{3.1}$$

$$((X_{t+m},\, t\ge 0),\mathbb{P}_x) \;=\; \left(\left(I^X\exp \underrightarrow{\xi}_{\underrightarrow{\tau}(t/I^X)},\, t\ge 0\right),\mathbf{P}\right), \tag{3.2}$$

*where* $\underrightarrow{\tau}(t)=\inf\left\{s:\int_0^s \exp\left(\underrightarrow{\xi}_u\right)\,du>t\right\}$, *for $t\ge 0$.*

*Proof.* The expression of the pre-minimum part of $(X,\mathbb{P}_x)$ follows directly from Lamperti's transformation (Section 2). Note that in particular, since $\tau$ is a continuous and strictly increasing function, one has:

$$A_\rho=\int_0^\rho \exp\xi_s\mathrm{d}s\,,\quad \tau(A_\rho)=\rho\,,\quad xA_\rho=m\quad\text{and}\quad I^X=x\exp I^\xi\,. \tag{3.3}$$

To express the post-minimum part of $(X,\mathbb{P}_x)$, first note that

$$X_{m+t}=x\exp\xi_{\tau(A_\rho+t/x)}\,,\quad t\ge 0\,.$$

Then we can write the time change as follows:

$$\begin{aligned}
\tau(A_\rho+t/x) &= \inf\{s>0:\int_0^s \exp\xi_u\,du>A_\rho+t/x\}\\
&= \inf\{s>\rho:\int_0^{s-\rho} \exp\xi_{u+\rho}\,du>t/x\}\\
&= \inf\{s>0:\int_0^{s} \exp\underrightarrow{\xi}_u\,du>(t/x)\exp(-I^\xi)\}+\rho\\
&= \underrightarrow{\tau}((t/x)\exp(-I^\xi))+\rho=\underrightarrow{\tau}(t/I^X)+\rho\,,
\end{aligned}$$

so that

$$\xi_{\tau(A_\rho+t/x)}=\xi_{\underrightarrow{\tau}(t/I^X)+\rho}=\underrightarrow{\xi}_{\underrightarrow{\tau}(t/I^X)}+I^\xi$$

and the expression (3.2) for the post-minimum part of $(X,\mathbb{P}_x)$ follows.

From (3.1), we see that $(X_t\,,t<m)$ is a measurable functional of $(\xi_t\,,t<\rho)$ and from (3.2), $(X_{m+t},\, t\ge 0)$ is a functional of $I^X$ and $\underrightarrow{\xi}$. Since $I^X=x\exp I^\xi$, the conditional independence follows from Theorem 1. $\square$

When $X$ has no positive jumps (or equivalently when $\xi$ has no positive jumps), it makes sense to define the last passage time at level $y\ge x$ as follows

$$\sigma_y=\sup\{t:X_t=y\}\,.$$

Then the post-minimum process of $X$ becomes more explicit as the following result shows; its proof is an easy consequence of Proposition 1.

**Proposition 2.** *Let $y \leq x$. Conditionally on $I^X = y$, the post-minimum process $(X_{t+m},\, t \geq 0)$ has the same law as $(X_{\sigma_y+t},\, t \geq 0)$, and*

$$((X_{\sigma_y+t},\, t \geq 0), \mathbb{P}_x) \stackrel{(d)}{=} \left( \left( y \exp \underset{\rightarrow}{\xi}_{\underset{\rightarrow}{T}(t/y)},\, t \geq 0 \right), \mathbf{P} \right). \tag{3.4}$$

As we have just seen, the post-minimum process of $(X, \mathbb{P})$ can be completely described using the underlying Lévy process $(\xi, \mathbf{P})$ conditioned to stay positive $(\underset{\rightarrow}{\xi}, \mathbf{P})$. Nevertheless, the description of the pre-minimum obtained in (3.1) is not so explicit. So our next purpose is to make some contributions to the understanding of the pre-minimum process of a positive self-similar Markov process.

Let us start by the case where the process $(X, \mathbb{P})$ (or equivalently the underlying Lévy process $(\xi, \mathbf{P})$) has no negative jumps, because in this case we can provide a more precise description of the pre-minimum process using known results for Lévy processes. Recall that the overall minimum of $(\xi, \mathbf{P})$, $-I^\xi$, follows an exponential law of parameter $\gamma > 0$ for some $\gamma$ which is determined in terms of the law $\mathbf{P}$. (See Bertoin [3], Chapter VII.) Furthermore, it has been proved by Bertoin [1] that the pre-minimum part of $(\xi, \mathbf{P})$ has the same law as a real valued Lévy process, say $(\xi, \mathbf{P}^{\downarrow})$, killed at its first hitting time of $-\mathfrak{e}$ with $\mathfrak{e}$ a r.v. independent of $(\xi, \mathbf{P}^{\downarrow})$ and that follows an exponential law of parameter $\gamma$. (The process $(\xi, \mathbf{P})$ can be viewed as $(\xi, pr)$ conditioned to drift to $-\infty$.) The translation of Bertoin's results for positive self-similar Markov process leads to the following Proposition. Denote by $\mathbb{P}^{\downarrow}$, the law of the process obtained by applying Lamperti's transformation to the Lévy process $(\xi, \mathbf{P}^{\downarrow})$.

**Proposition 3.** *If $(X, \mathbb{P})$, equivalently $(\xi, \mathbf{P})$, has no negative jumps, then there exists a real $\gamma > 0$ such that for any $x > 0$*

$$\mathbb{P}_x(I^X \leq \epsilon) = (\epsilon/x)^\gamma \wedge 1, \quad \epsilon \geq 0,$$

*and the law of $((X_t, 0 \leq t < m), \mathbb{P}_x)$ is the same as that of $((X_t, 0 \leq t < T(Z)), \mathbb{P}^{\downarrow}_x)$, where $Z$ is a random variable independent of $(X, \mathbb{P}^{\downarrow}_x)$ and such that $(-\log(Z/x), \mathbb{P}^{\downarrow}_x)$ follows an exponential law of parameter $\gamma > 0$.*

*Proof.* This follows from Proposition 1 and Theorem 2 in [1], described above. □

So to reach our end, we will next provide a description of the pre-minimum of a real valued Lévy process that drifts to $\infty$, which generalizes Bertoin's result and is analogous to the description of the pre-minimum of a Lévy process conditioned to stay positive that has been obtained in [9] and [17].

Let $\widehat{V}(\mathrm{d}x), x \geq 0$ be the renewal measure of the downward ladder height process, see e.g. [3] or [13] for background. In the remaining of this Section we will assume that under $\mathbf{P}$,

$$\text{(H)} \quad \begin{cases} 0 \text{ is regular for } ]-\infty, 0[ \\ \xi \text{ derives towards } +\infty \\ \text{the measure } \widehat{V}(\mathrm{d}x) \text{ is absolutely continuous w.r.t Lebesgue's measure.} \end{cases}$$

In order to construct the Lévy process which describes the pre-minimum part of $(\xi, \mathbf{P})$ we will need the following Lemma which is reminiscent of Theorem 1 in [28]. Let $\mathbf{P}^{]-\infty,0[}$ be the law of $(\xi, \mathbf{P})$ killed at its first hitting time of $]-\infty, 0[$.

**Lemma 1.** *Under the assumptions $(H)$ the renewal measure $\widehat{V}(\mathrm{d}x)$ has a density, say $\varphi : \mathbb{R} \to \mathbb{R}^+$, which is excessive for the semigroup of $(\xi, \mathbf{P}^{]0,\infty[})$ and $0 < \varphi(x) < \infty$ for a.e. $x \in \mathbb{R}^+$.*

*Proof.* It is known that the processes $(\xi, \mathbf{P})$ and $(-\xi, \mathbf{P})$ are in weak duality w.r.t. Lebesgue's measure, so by Hunt's switching identity we have that $(\xi, \mathbf{P}^{]-\infty,0[})$ and $(-\xi, \mathbf{P}^{]-\infty,0[})$ are also in weak duality w.r.t. Lebesgue measure, see e.g. [20]. On the other hand, it is known that the measure $\widehat{V}(dx)$ is an invariant measure for the process, $S - \xi = \{\sup_{s\leq t} \xi_s - \xi_t, t \geq 0\}$, $\xi$ reflected at its supremum, see e.g. [3] Chapter VI exercise 5. So the measure, $\widehat{V}(\mathrm{d}x)$ is excessive for $S - \xi$, killed at its first hitting time of 0, so for $(-\xi, \mathbf{P}^{]-\infty,0[})$. Thus the first assertion of Lemma 1 is a direct consequence of Theorem in Chapter XII paragraph 71 in [14]. To prove the second assertion we recall that

$$\widehat{V}[0, x] = k\,\mathbf{P}(-\inf_{0\leq s<\infty} \xi_s \leq x), \qquad x \geq 0,$$

with $k \in ]0, \infty[$ a constant, see [3] Proposition VI.17. So $\varphi < \infty$ a.e. and by the regularity for $]-\infty, 0[$ of 0, the support of the law of $\inf_{0\leq s<\infty} \xi_s$ is $]-\infty, 0[$, thus $0 < \varphi$ a.e. $\square$

Let $\mathbf{P}^{\searrow}$, be the $h$-transform of the law, $\mathbf{P}^{]-\infty,0[}$, via the excessive function $\varphi$. That is, $\mathbf{P}^{\searrow}$ is the unique measure which is carried by $\{0 < \zeta\}$ and under which the canonical process is Markovian with semi-group $(P_t^{\searrow}, t \geq 0)$,

$$P_t^{\searrow} f(x) = \begin{cases} \frac{1}{\varphi(x)}\, \mathbf{E}_x^{]-\infty,0[}\left(f(\xi_t)\varphi(\xi_t)\right) & \text{if } x \in \{z \in \mathbb{R} : 0 < \varphi(z) < \infty\}, \\ 0 & \text{if } x \notin \{z \in \mathbb{R} : 0 < \varphi(z) < \infty\}. \end{cases}$$

Let $\Lambda = \{z \in \mathbb{R} : 0 < \varphi(z) < \infty\}$. Furthermore, the measure $\mathbf{P}^{\searrow}$ is carried by $\{\xi_t \in \Lambda, \xi_{t-} \in \Lambda, t \in ]0, \zeta[\}$, and for any $\mathcal{G}_t$-stopping time $T$

$$\mathbf{P}_x^{\searrow}\, 1_{\{T<\zeta\}} = \frac{\varphi(\xi_T)}{\varphi(x)} 1_{\{T<\zeta\}}\, \mathbf{P}_x^{]-\infty,0[}, \qquad \text{on } \mathcal{G}_T.$$

In the case where the semigroup of $(\xi, \mathbf{P})$ is absolutely continuous, $\mathbf{P}^{\searrow}$ has been introduced in [9] where it is proved that this measure can be viewed as the law of $(\xi, \mathbf{P})$ conditioned to hit 0 continuously. In the case where $(\xi, \mathbf{P})$ creeps downward $\varphi$ can be made explicit:

$$\varphi(x) = c\,\mathbf{P}(\xi_{T_{]-\infty,-x[}} = -x) > 0, \qquad x > 0,$$

with $0 < c < \infty$, a constant, see [3] Theorem VI.19, and then we have the right conditioning:

$$\mathbf{P}_x^{\searrow} = \mathbf{P}_x^{]-\infty,0[}(\,\cdot \mid \xi_{T_{]-\infty,0[}} = 0).$$

So in the sequel we will refer to $\mathbf{P}^{\searrow}$ as the law of $(\xi, \mathbf{P})$ conditioned to hit 0.

**Lemma 2.** *Let $\xi$ be a real valued Lévy process that satisfies the hypotheses $(H)$ and $\varphi$ be the density of the renewal measure $\widehat{V}$ as in Lemma 1. Then for any bounded measurable functional $F$,*

$$\mathbf{E}\left(F\left(\xi_s-\xi_{\rho-},0\leq s<\rho\right)\right)=\frac{1}{\widehat{V}]0,\infty[}\int_{]0,\infty[}\mathrm{d}a\varphi(a)\,\mathbf{E}^{\searrow}_a\left(F\left(\xi_s,0\leq s<\zeta\right)\right).$$

*In particular under $\mathbf{P}$ conditionally on $I^{\xi}=a$, the pre-minimum process has the same law as $\xi+a$ under $\mathbf{P}^{\searrow}_{-a}$.*

Observe that Bertoin's [1] Theorem 2 can be deduced from this Lemma since in the case where $\xi$ has no negative jumps $\varphi$, is given by $\varphi(x)=\gamma e^{-\gamma x}, x>0$, and so we have that

$$\begin{aligned}
\mathbf{E}\left(F\left(\xi_s,0\leq s<\rho\right)\right)&=\int_{]0,\infty[}\mathrm{d}a\gamma e^{-\gamma a}\,\mathbf{E}^{\searrow}_a\left(F\left(\xi_s-a,0\leq s<\zeta\right)\right)\\
&=\int_{]0,\infty[}\mathrm{d}a\gamma e^{-\gamma a}\,\mathbf{E}^{]-\infty,0[}_a\left(F\left(\xi_s-a,0\leq s<\zeta\right)\mid T_{]-\infty,0[}<\infty\right)\\
&=\gamma\int_{]0,\infty[}\mathrm{d}a\,\mathbf{E}\left(F\left(\xi_s,0\leq s<T_{]-\infty,-a[}\right),T_{]-\infty,-a[}<\infty\right)\\
&=\int_{]0,\infty[}\mathrm{d}a\gamma e^{-\gamma a}\,\mathbf{E}\left(F\left(\xi_s,0\leq s<T_{]-\infty,-a[}\right)e^{-\gamma\xi_{T_{]-\infty,-a[}}},\ T_{]-\infty,-a[}<\infty\right)\\
&=\mathbf{E}^{\downarrow}\left(F\left(\xi_s,0\leq s<T_{]-\infty,-\mathfrak{e}[}\right)\right),
\end{aligned}$$

where $\mathbf{P}^{\downarrow}$ and $\mathfrak{e}$ are as explained just before Proposition 3.

*Proof.* To prove the claimed identity, we will start by calculating for any continuous and bounded functional $F$,

$$\mathbf{E}^{\mathrm{e}/\lambda}\left(F\left(\xi_s-\xi_{\rho},0\leq s<\rho\right)\right),$$

where $\mathbf{E}^{\mathrm{e}/\lambda}$ is the law of $(\xi,\mathbf{P})$ killed at time $\mathrm{e}/\lambda$, with e an exponential random variable independent of $(\xi,\mathbf{P})$. To do that we will denote by $\{\underline{L}_u,u\geq 0\}$ the local time at 0 of the strong Markov process $\{\xi_t-I_t,t\geq 0\}$, by $g_t$ the last hitting time of 0 by $\xi-I$ before time $t$, $g_t=\sup\{s\leq t:\xi_s-I_s=0\}$, and by $\underline{N}$ the excursion measure of $\xi-I$ away from 0. Indeed, using Maisonneuve's exit formula of excursion theory it is justified that

$$\begin{aligned}
\mathbf{E}^{\mathrm{e}/\lambda}\left(F\left(\xi_s-I_{\rho},0\leq s<\rho\right)\right)&=\int_0^{\infty}\mathrm{d}t\lambda e^{-\lambda t}\,\mathbf{E}\left(F\left(\xi_s-I_{g_t},0\leq s<g_t\right)\right)\\
&=\int_0^{\infty}\mathrm{d}t\lambda e^{-\lambda t}\,\mathbf{E}\left(\int_0^t\mathrm{d}\underline{L}_uF\left(\xi_s-I_u,0\leq s<u\right)\underline{N}(t-u<\zeta)\right)\\
&=\mathbf{E}\left(\int_0^{\infty}\mathrm{d}\underline{L}_ue^{-\lambda u}F\left(\xi_s-I_u,0\leq s<u\right)\right)\underline{N}(1-e^{-\lambda\zeta}).
\end{aligned}$$

Next, making $\lambda$ tend to 0, the left hand term in the previous equality tends to

$$\mathbf{E}\left(F\left(\xi_s-\xi_{\rho-},0\leq s<\rho\right)\right),$$

while the right hand term tends to

$$\mathbf{E}\left(\int_0^{\infty}\mathrm{d}\underline{L}_uF\left(\xi_s-I_u,0\leq s<u\right)\right)\underline{N}(\zeta=\infty).$$

Finally, a straightforward extension of Lemma 3 in [10] to our weaker hypothesis allows us to ensure that

$$\mathbf{E}\left(\int_0^\infty \mathrm{d}\underline{L}_u F\left(\xi_s - I_u, 0 \leq s < u\right)\right) = \int_{]0,\infty[} \mathrm{d}a\varphi(a)\, \mathbf{E}_a^{\searrow}\left(F\left(\xi_s, 0 \leq s < \zeta\right)\right),$$

which concludes the proof given that $\widehat{V}]0,\infty[= (\underline{N}(\zeta = \infty))^{-1}$. □

We next introduce the law of a Lévy process conditioned to hit by above a given level $a \in \mathbb{R}$. Owing to the fact that the function $\varphi$ is excessive for $(\xi, \mathbf{P})$ killed at 0, we have that for any $a \in \mathbb{R}$ the function $\varphi_a : \mathbb{R} \to \mathbb{R}^+$ defined by $\varphi_a(x) = \varphi(x-a), x \in \mathbb{R}$, is excessive for the semigroup of $\xi$ killed at its first hitting time of $]-\infty, a[$. Indeed,

$$\mathbf{E}_x(\varphi_a(\xi_t), t < T_{]-\infty,a[}) = \mathbf{E}_{x-a}(\varphi(\xi_t), t < T_{]-\infty,0[}) \leq \varphi(x-a) = \varphi_a(x), \qquad x > a,$$

and analogously it is verified that $\lim_{t\to 0+} \mathbf{E}_x(\varphi_a(\xi_t), t < T_{]-\infty,a[}) = \varphi_a(x)$. We will denote by $\mathbf{P}^{\searrow a}$ the $h$-transform of the law of $\xi$ killed at it first hitting time of $]-\infty, a[$ via $\varphi_a$, i.e.: for any $\mathcal{G}_t$-stopping time $T$, with an obvious notation for $\mathbf{P}_x^{]-\infty,a[}$,

$$\mathbf{P}_x^{\searrow a}\, 1_{\{T<\zeta\}} = \frac{\varphi_a(\xi_T)}{\varphi_a(x)} 1_{\{T<\zeta\}}\, \mathbf{P}_x^{]-\infty,a[}, \qquad \text{on } \mathcal{G}_T.$$

The following elementary Lemma will enable us to refer to this measure as the law of $\xi$ conditioned to hit $a$ continuously and by above. Of course the measure $\mathbf{P}_x^{\searrow 0}$ is simply $\mathbf{P}_x^{\searrow}$.

**Lemma 3.** *Let $(\xi, \mathbf{P})$ be a real valued Lévy process that satisfies the hypotheses $(H)$. For $a \in \mathbb{R}$, and any $x > a$ the law of $\xi + a$ under $\mathbf{P}_{x-a}^{\searrow}$ is the same as that of $\xi$ under $\mathbf{P}_x^{\searrow a}$. As a consequence, for a.e. $x > a$*

$$\mathbf{P}_x^{\searrow a}\left(\xi_0 = x;\ \zeta < \infty;\ \xi_t > a \text{ for all } t < \zeta;\ \xi_{\zeta-} = a\right) = 1.$$

*Proof.* To prove the first assertion it suffices to verify that both laws are equal over $\mathcal{G}_t$ for any $t > 0$. Indeed, the spatial homogeneity of $(\xi, \mathbf{P})$ implies that for $t > 0$ and any bounded measurable functional $F$

$$\begin{aligned}\mathbf{P}_x^{\searrow a}\left(F(\xi_s, 0 \leq s < t) 1_{\{t<\zeta\}}\right) &= \frac{1}{\varphi_a(x)} \mathbf{P}_x(F(\xi_s, 0 \leq s < t) 1_{\{t<T_{]-\infty,a[}\}} \varphi_a(\xi_t)) \\ &= \frac{1}{\varphi(x-a)} \mathbf{P}_{x-a}(F(\xi_s + a, 0 \leq s < t) 1_{\{t<T_{]-\infty,0[}\}} \varphi(\xi_t)) \\ &= \mathbf{P}_{x-a}^{\searrow}(F(\xi_s + a, 0 \leq s < t) 1_{\{t<\zeta\}}).\end{aligned}$$

Now, the second assertion is an easy consequence of Lemma 2 and the hypothesis that 0 is regular for $]-\infty, 0[$. □

A rewording of Lemma 2 using Lemma 3 reads:

**Theorem 2.** *Let $(\xi, \mathbf{P})$ be a real valued Lévy process that satisfies the hypotheses $(H)$. The following identity holds for any bounded measurable functional $F$,*

$$\mathbf{E}_x\left(F\left(\xi_s, 0 \leq s < \rho\right)\right) = \frac{1}{\widehat{V}]0,\infty[} \int_{]-\infty,x[} \mathrm{d}a\varphi_a(x)\, \mathbf{E}_x^{\searrow a}\left(F\left(\xi_s, 0 \leq s < \zeta\right)\right). \tag{3.5}$$

We have now all the elements to state the main result of this section whose proof follows easily from Lemma 3 & Theorem 2.

**Theorem 3.** *Let $(\xi, \mathbf{P})$ be a real valued Lévy process that satisfies the hypotheses $(H)$ and $(X, \mathbb{P})$ be the self-similar Markov process associated to $(\xi, \mathbf{P})$ via Lamperti's representation. Then for any bounded measurable functional $F$,*

$$\begin{aligned}\mathbb{E}_x\left(F\left(X_s, 0 \leq s < m\right)\right) &= \int_0^1 \nu_1(\mathrm{d}v)\, \mathbb{P}_x^{\searrow vx}(F(X_s, 0 \leq s < \zeta)) \\ &= \int_0^1 \nu_1(\mathrm{d}v)\, \mathbb{P}_{1/v}^{\searrow 1}(F(vxX_{s/vx}, 0 \leq s < vx\zeta)),\end{aligned}$$

*where $\nu_1$ is a measure over $]0,1[$ with density*

$$\frac{\nu_1(\mathrm{d}v)}{\mathrm{d}v} = (\widehat{V}]0,\infty[)^{-1} v^{-1} \varphi(-\ln v), \qquad 0 < v < 1,$$

*and $\mathbb{P}_x^{\searrow vx}$ is the law of the process obtained by applying Lamperti's representation to $(\xi,\, \mathbf{P}_{\log(x)}^{\searrow \log(vx)})$.*

# 4 The asymptotic behavior of the pre- and post-minimum as the minimum tends to 0.

Throughout this section we will leave aside the assumptions $(H)$. We only assume that the underlying Lévy process $\xi$ drifts to $\infty$, it is not a subordinator and it is non lattice. Some ancillary hypothesis will be stated below.

## 4.1 Post-minimum

Under these hypotheses, it is known that the support of the law of $I^\xi$ is $]-\infty, 0]$. From (3), the support of $I^X$ is then $[0, y]$ under $\mathbb{P}_y$, for any $y > 0$. Proposition 1 shows that a regular version of the law of the post-minimum process $(X_{m+t},\, t \geq 0)$ under $\mathbb{P}_y$ given $I^X = x$, for $x \in ]0, y]$ is given by the law of the process $\left(\left(x\, \underset{\rightarrow}{\xi}_{\underset{\rightarrow}{\tau}(t/x)},\, t \geq 0\right), \mathbf{P}\right)$. In particular, this law does not depend on $y$. Let us denote it by $\underset{\rightarrow}{\mathbb{P}}^x$. A straight consequence of this representation is that the family $(\underset{\rightarrow}{\mathbb{P}}^x)$ is weakly continuous on $]0, \infty[$. In Theorem 4 below, we show that if moreover $0 < \mathbf{E}(\xi_1) < \infty$, then $\underset{\rightarrow}{\mathbb{P}}^x$ converges weakly as $x$ tends to 0 towards the law $\mathbb{P}_{0+}$. This measure is the weak limit of $\mathbb{P}_x$ as $x \to 0+$, whose existence is ensured by Theorem 2 in [11].

Recall that Millar's results implies that for any $x > 0$, the process $(X, \underset{\rightarrow}{\mathbb{P}}^x)$ is strongly Markov with values in $[x, \infty[$.

**Theorem 4.** *Assume that $0 < \mathbf{E}(\xi_1) < \infty$. The laws $\underset{\rightarrow}{\mathbb{P}}^x$ converge weakly in $\mathbb{D}$ as $x \to 0+$ to the law $\mathbb{P}_{0+}$. As a consequence, for any $x > 0$,*

$$\mathbb{P}_x(\cdot \circ \theta_m | I^X < \epsilon) \xrightarrow[\epsilon \to 0]{w} \mathbb{P}_{0+}(\cdot).$$

*Proof.* Recall that from [11], under our hypothesis, the family of laws $(\mathbb{P}_x)$ converges weakly in $\mathbb{D}$ as $x \downarrow 0$ towards the non degenerate law of a self-similar strong Markov process. Denote by $\mathbb{P}_{0+}$ the limit law. Then on the space $\mathbb{D}$, we define a process $X^{(0)}$ with law $\mathbb{P}_{0+}$. We recall from [11] that

$$\lim_{t\to 0+} X_t^{(0)} = 0 \quad \text{and} \quad \lim_{t\uparrow\infty} X_t^{(0)} = +\infty\,, \quad \mathbb{P}_{0+} \text{ a.s.} \tag{4.1}$$

Let $(x_n)$ be any sequence of positive real numbers which tends to 0. Define $\Sigma_n = \inf\{t : X_t^{(0)} \ge x_n\}$, then by the Markov property and Lamperti's representation, we have

$$Y^{(n)} \stackrel{(\mathrm{def})}{=} (X^{(0)}_{\Sigma_n+t},\, t\ge 0) = \left(X^{(0)}_{\Sigma_n} \exp \xi^{(n)}_{\tau^{(n)}(t/X^{(0)}_{\Sigma_n})},\, t\ge 0\right), \tag{4.2}$$

where on the left hand side of the second equality, $X^{(0)}_{\Sigma_n}$ and $\xi^{(n)}$ are independent and $\xi^{(n)} \stackrel{(d)}{=} \xi$. Let

$$I_n = \inf_{t\ge 0} Y_t^{(n)} \quad \text{and} \quad m_n = \sup\{t : Y_t^{(n)} \wedge Y_{t-}^{(n)} = I_n\}\,.$$

Then we deduce from (4.2) and Proposition 1 the following representation:

$$(Y^{(n)}_{m_n+t},\, t\ge 0) = \left(I_n \exp \underset{\rightarrow}{\xi}^{(n)}_{\underset{\rightarrow}{\tau}(t/I_n)},\, t\ge 0\right), \tag{4.3}$$

where $\underset{\rightarrow}{\xi}^{(n)}$ is independent of the events prior to $m_n$. In particular, $\underset{\rightarrow}{\xi}^{(n)}$ is independent of $\mathcal{G}_n \stackrel{(\mathrm{def})}{=} \sigma\{I_k : k\ge n\}$. It follows from (4.3) that for any bounded and measurable functional $H$,

$$\mathbb{E}_{0+}(H(Y^{(n)}_{m_n+t},\, t\ge 0)\,|\,\mathcal{G}_n) = \underset{\rightarrow}{\mathbb{E}}^{I_n}(H)\,. \tag{4.4}$$

Since $(X, \mathbb{P}_x)$, $x\ge 0$ is a Feller process, the tail $\sigma$-field $\cap_{t>0}\sigma\{X_s^{(0)} : s\le t\}$ is trivial and it is not difficult to check that $\cap_n \mathcal{G}_n \subset \sigma\{X_s^{(0)} : s\le t\}$ for each fixed $t$. So $\cap_n\mathcal{G}_n$ is trivial. On the other hand, from (4.1) we have $\lim_n \Sigma_n = 0$ and $\lim_n m_n = 0$, $\mathbb{P}_{0+}$–a.s., so

$$(Y^{(n)}_{m_n+t},\, t\ge 0) \longrightarrow X^{(0)}\,, \quad \mathbb{P}_{0+} \text{ a.s., as } n\to+\infty,$$

on the space $\mathbb{D}$. Hence if we suppose moreover that $H$ is continuous, then

$$\lim_n E(H(Y^{(n)}_{m_n+t},\, t\ge 0)\,|\,\mathcal{G}_n) = \lim_n \underset{\rightarrow}{\mathbb{E}}^{I_n}(H) = \mathbb{E}_0(H)\,, \quad \mathbb{P}_{0+} \text{ almost surely}\,. \tag{4.5}$$

Now, from (3.3), we have $I_n = X^{(0)}_{\Sigma_n} \exp I^{\xi^{(n)}}$. Recall that from Theorem 1 in [11], the r.v. $X^{(0)}_{\Sigma_n}$ may be decomposed as $X^{(0)}_{\Sigma_n} \stackrel{(d)}{=} x_n e^{\theta}$, where $\theta$ is a finite r.v. whose law is this of the limit overshoot of the Lévy process $\xi$, i.e. if $T_z = \inf\{t : \xi_t \ge z\}$, then under our hypothesis, $\xi_{T_z} - z$ converges in law as $z\uparrow+\infty$ towards the law of $\theta$. So, we have

$$I_n \stackrel{(d)}{=} x_n e^{\theta} e^{I^{\xi}}\,, \tag{4.6}$$

where $\theta$ and $I^{\xi}$ are independent. On the space $\mathbb{D}$, we define a r.v. $\nu$ such that $\nu \stackrel{(d)}{=} e^{\theta}e^{I^{\xi}}$ (so that $I_n \stackrel{(d)}{=} x_n\nu$), then it follows from (4.5) that

$$\underset{\rightarrow}{\mathbb{E}}^{x_n\nu}(H) \longrightarrow \mathbb{E}_0(H)\,, \quad \text{in probability, as } n\to+\infty\,. \tag{4.7}$$

So there exists a subsequence $x_{n_k}$ such that

$$\underset{\rightarrow}{\mathbb{E}}^{x_{n_k}\nu}(H) \longrightarrow \mathbb{E}_0(H)\,, \quad \text{a.s., as } k \to +\infty\,. \tag{4.8}$$

The convergence (4.8) implies that there exists $\omega_0 \in \mathbb{D}$ such that $\nu(\omega_0) > 0$ and $\underset{\rightarrow}{\mathbb{E}}^{x_{n_k}\nu(\omega_0)}(H) \to \mathbb{E}_0(H)$, as $k \to +\infty$. Put $a = \nu(\omega_0)$ and for all $\omega \in \mathbb{D}$ define $S_a(\omega) = (a^{-1}\omega_{at},\, t \geq 0)$. Since $S_a$ is a continuous functional on $\mathbb{D}$, we have

$$\underset{\rightarrow}{\mathbb{E}}^{x_{n_k}a}(H \circ S_a) \longrightarrow \mathbb{E}_0(H \circ S_a)\,, \quad \text{as } k \to +\infty\,.$$

But from the scaling property, we have $\underset{\rightarrow}{\mathbb{E}}^{x_{n_k}a}(H \circ S_a) = \underset{\rightarrow}{\mathbb{E}}^{x_{n_k}}(H)$ and $\mathbb{E}_{0+}(H \circ S_a) = \mathbb{E}_{0+}(H)$. In conclusion, for any bounded and continuous functional $H$ on $\mathbb{D}$ and for any sequence $(x_n)$ which decreases to 0, there is a subsequence $(x_{n_k})$ such that $\underset{\rightarrow}{\mathbb{E}}^{x_{n_k}}(H) \longrightarrow \mathbb{E}_{0+}(H)$, as $k$ tends to $\infty$. This proves our result. □

## 4.2 Pre-minimum

In our description of the pre-minimum process we have provided, under some assumptions, a method to construct a process that can be viewed as $X$ conditioned to die at a given level $0 < a < X_0$. But a priori this method cannot be applied to construct a process that dies at 0, since this means conditioning the underlying Lévy process to die at $-\infty$. Thus, the purpose of this section is to construct the law of a self-similar Markov process that can be viewed as the law of a pssMp that drift to $\infty$, $X$, conditioned to hit 0 in a finite time. In fact we will answer the questions: *What is the process obtained by making tend to* 0 *the value of the overall minimum of* $X$*? Is the resulting process determined by the pre-minimum process of* $X$*?* In the case where $X$ has no negative jumps, using the assertions in Proposition 3 it is clear, at least intuitively, that the process $(X, \mathbb{P}^{\downarrow})$ can be obtained from $(X, \mathbb{P})$ by making tend to 0 the value of its overall minimum. Actually, the former process can be viewed as $(X, \mathbb{P})$ conditioned to have an overall minimum equal to 0 and this conditioning depends only on the pre-minimum part of $(X, \mathbb{P})$.

As a consequence of the assumption that $(X, \mathbb{P})$ drifts to $\infty$, the set of paths that have an overall minimum equal to 0, $\{I^X = 0\}$, has probability 0, and so the law of $(X, \mathbb{P})$ conditionally on that set does not make sense. A natural issue to give a meaning to that conditioning is by approximating that set by the sequence $\{I^X < \epsilon\}$ as $\epsilon \to 0$. So, our main task will be describe the limit law of the pre-minimum process conditionally on the event $\{I^X < \epsilon\}$ as $\epsilon \to 0$. To that end we will use the method of $h$-transformations.

Let $h : ]0, \infty[ \to [0, \infty]$ be the function defined by

$$h(x) = \liminf_{\epsilon \to 0} \frac{\mathbb{P}_x(I^X < \epsilon)}{\mathbb{P}_1(I^X < \epsilon)}, \qquad x \in ]0, \infty[. \tag{4.9}$$

The following Lemma will be useful.

**Lemma 4.** *The function* $h$ *defined in equation* (4.9) *is excessive for the semigroup of the pssMp* $X$.

*Proof.* Given that the cone of excessive functions is closed under *liminf* it suffices with proving that for every $\epsilon > 0$, the function

$$h^{\epsilon}(x) = \frac{\mathbb{P}_x(I^X < \epsilon)}{\mathbb{P}_1(I^X < \epsilon)}, \qquad x \in ]0, \infty[,$$

is excessive for the semigroup of $X$. Indeed, owing the relation

$$\mathbb{P}_x(I^X < \epsilon) = \mathbb{P}_x(L_{\epsilon} > 0), \qquad \text{with } L_{\epsilon} = \sup\{s > 0 : X_s < \epsilon\}, \ (\sup\{\emptyset\} = 0),$$

and the Markov property, it is straightforward that for any reals $t > 0$ and $x > 0$

$$P_t h^{\epsilon}(x) = \frac{\mathbb{E}_x(\mathbb{P}_{X_t}(L_{\epsilon} > 0))}{\mathbb{P}_1(I^X < \epsilon)} = \frac{\mathbb{P}_x((L_{\epsilon} - t)^+ > 0)}{\mathbb{P}_1(I^X < \epsilon)} \leq h^{\epsilon}(x), \qquad x > 0,$$

and

$$\lim_{t \to 0} P_t h^{\epsilon}(x) = h^{\epsilon}(x), \qquad x > 0.$$

□

To perform the desired conditioning we will make some assumptions on the excessive function $h$. Firstly, to avoid pathological cases we will assume that $h$ does not take the values 0 or $\infty$, and next that it has some regularity, namely that

**(H')** the liminf in equation (4.9) is in fact a limit and $h : ]0, \infty[ \to ]0, \infty[$ is a non-constant function.

The hypothesis (H') is satisfied by a wide class of positive self-similar Markov processes, as it will be seen in Remark 1 below, and, whenever it holds, the self-similarity implies that, the excessive function $h$ has the form

$$h(x) = x^{-\gamma}, \qquad x > 0, \text{ for some } \gamma > 0.$$

Here is a reformulation of (H') in terms of the underlying Lévy process $(\xi, \mathbf{P})$. First, one has $\mathbf{P}(-I^{\xi} > z) > 0$ for each $z > 0$ and

$$\lim_{u \to \infty} \frac{\mathbf{P}(-I^{\xi} > u - z)}{\mathbf{P}(-I^{\xi} > u)} = e^{\gamma z} \quad \text{for each } z \in \mathbb{R}\,.$$

In other words, the law of the negative of the overall minimum of $\xi$ belongs to one of the classes $\mathcal{L}^{\gamma}$, for some $\gamma > 0$;

In the sequel we will assume that the hypothesis (H') is satisfied. Let $\mathbb{P}^{\downarrow}$ be the $h$-transform measure of $\mathbb{P}$ via $h$, i.e.: for any $\mathcal{F}_t$-stopping time $T$

$$\mathbb{P}^{\downarrow}_x 1_{\{T < \zeta\}} = \frac{h(X_T)}{h(x)} \mathbb{P}_x, \qquad \text{on } \mathcal{F}_T.$$

By standard arguments it follows that the law $\mathbb{P}^{\downarrow}$ is that of a positive self-similar Markov process, say $(X, \mathbb{P}^{\downarrow})$. We will denote by $(\xi, \mathbf{P}^{\downarrow})$ the Lévy process associated to $(X, \mathbb{P}^{\downarrow})$ via

Lamperti's transformation. By the absolute continuity relation between $\mathbb{P}^{\downarrow}$ and $\mathbb{P}$ applied to the sequence of $\mathcal{F}$-stopping times

$$T_t = \inf\{r > 0 : \int_0^r X_s^{-1} \mathrm{d}s > t\}, \qquad t \geq 0,$$

and Lamperti's transformation, it holds that $\mathbf{E}(e^{-\gamma\xi_t}) \leq 1$ for all $t > 0$, and more importantly that the laws $\mathbf{P}^{\downarrow}$ and $\mathbf{P}$ are absolutely continuous: for any $t \geq 0$

$$\mathbf{P}^{\downarrow} 1_{\{t<\zeta\}} = e^{-\gamma\xi_t} \mathbf{P}, \quad \text{on } \mathcal{F}_{T_t} = \mathcal{G}_t. \tag{4.10}$$

The latter relation can be extended to $\mathcal{G}$-stopping times using standard arguments.

With the following result we prove that the family of laws $(\mathbb{P}^{\downarrow}_x, x > 0)$ can be thought as those of the process $(X, \mathbb{P})$ strictly before $m$ when the whole trajectory is conditioned to have an overall minimum equal to 0.

**Theorem 5.** *Assume the hypothesis $(H')$ is satisfied.*

$(i)$ *The process $(X, \mathbb{P}^{\downarrow}_x)$ hits 0 in a finite time, a.s. Moreover,*

$$\mathbb{P}^{\downarrow}_x (T_0 < \infty, X_{T_0-} = 0) = 1, \quad \textit{for all } x > 0,$$

*if and only if Cramér's condition, $\mathbb{E}(e^{-\gamma\xi_1}) = 1$, is satisfied.*

$(ii)$ *If $(\xi, \mathbf{P})$ satisfies furthermore that either*

$(a1)$ *its law is not lattice,*

$(a2)$ *Cramér's condition, $\mathbb{E}(e^{-\gamma\xi_1}) = 1$ and $\mathbb{E}(\xi_1^- e^{-\gamma\xi_1}) < \infty$ are satisfied,*

*or*

$(b1)$ $\mathbb{E}(e^{-\gamma\xi_1}) < 1$,

*then the law $\mathbb{P}^{\downarrow}$ is determined by the law of the pre–minimum process of $(X, \mathbb{P})$ in the following way: for any $x > 0$*

$$\lim_{\epsilon \to 0+} \mathbb{P}_x(F_t \cap \{t < m\} | I^X < \epsilon) = \mathbb{P}^{\downarrow}_x(F_t \cap \{t < T_0\}), \quad F_t \in \mathcal{F}_t, \ t \geq 0.$$

A consequence of (ii) in Theorem 5 is that the finite dimensional laws of the pre-minimum process converge to those of $(X, \mathbb{P}^{\downarrow}_x)$.

*Proof of part $(i)$.* By the identity (4.10) it follows that

$$\mathbf{E}(e^{-\gamma\xi_t}) = \mathbf{P}^{\downarrow}(t < \zeta), \qquad \text{for all } t > 0,$$

and so under $\mathbf{P}^{\downarrow}$ the canonical process $\xi$ has an infinite lifetime if and only if $\mathbf{E}(e^{-\gamma\xi_t}) = 1$, for all $t > 0$ or equivalently for some $t > 0$, see e.g. Sato [27] Theorem 25.17. In which case Cramér's condition is satisfied and the process $(\xi, \mathbf{P}^{\downarrow})$ drifts to $-\infty$. Given that the process $(X, \mathbb{P}^{\downarrow})$

coincides with the pssMp associated to $(\xi, \mathbf{P}^{\downarrow})$ via Lamperti's transformation, we conclude using Lamperti's representation of pssMp, see Section 2, that if Cramér's condition is satisfied then

$$\mathbb{P}^{\downarrow}_x(T_0 < \infty, X_{T_0-} = 0) = 1, \quad \text{for all } x > 0.$$

Now, assume that Cramér's condition is not satisfied, that is $\mathbf{E}(e^{-\gamma\xi_t}) < 1$ for some $t > 0$. By Theorem 25.17 in [27] this implies that the latter holds for all $t > 0$. So the Lévy process $(\xi, \mathbf{P})$ has a finite lifetime, actually it is a real valued Lévy process that has been killed at an independent time that follows an exponential law of parameter $\kappa = -\log \mathbf{E}(e^{-\gamma\xi_1})$. According to Lamperti representation of pssMp we have that in this case

$$\mathbb{P}^{\downarrow}_x(T_0 < \infty, X_{T_0-} > 0) = 1, \quad \text{for all } x > 0.$$

In any case, $(X, \mathbb{P}^{\downarrow}_x)$ hits 0 in a finite time a.s. for all $x > 0$. Which finish the proof of assertion (i).

*Proof of part* (*ii*). To prove the assertion we will start by proving that for any $x > 0$ and $t > 0$,

$$\lim_{\epsilon\to 0+} \mathbb{P}_x(t < m | I^X < \epsilon) = x^{\gamma}\, \mathbb{E}_x(X_t^{-\gamma}) = \mathbb{P}^{\downarrow}_x(t < T_0). \tag{4.11}$$

To that end we will use that $\{L_\epsilon > 0\} = \{I^X < \epsilon\}$, and so that

$$\begin{aligned} \mathbb{P}_x\left(t < m, I^X < \epsilon\right) &= \mathbb{P}_x\left(t < m, 0 < L_\epsilon, t < L_\epsilon\right) \\ &= \mathbb{P}_x\left(m \wedge L_\epsilon > t\right) \\ &= \mathbb{P}_x\left(\mathbb{P}_{X_t}\left(m \wedge L_\epsilon > 0\right)\right) \\ &= \mathbb{P}_x\left(\mathbb{P}_{X_t}\left(L_\epsilon > 0\right)\right), \end{aligned}$$

which is a consequence of the fact that $L_\epsilon$ and $m$ are both coterminal times, the Markov property and that $\mathbb{P}_x(m = 0) = 0$, owing that $(\xi, \mathbf{P})$ is not a subordinator. Moreover, it follows from the scaling and Markov properties that

$$\mathbb{E}_x\left(\mathbb{P}_{X_t}\left(L_\epsilon > 0\right)\right) = \mathbb{E}_x\left(g(X_t/\epsilon)\right),$$

where $g(z) = \mathbb{P}_x\left(I^X \leq z^{-1}\right)$. Now, if the conditions (a-1,2) are satisfied, then the main result of [5] implies that $g(z) = z^{\gamma}L(z), z > 0$, where $L : ]0, \infty[\to]0, \infty[$ is a bounded and slowly varying function such that $L(z) \to C \in ]0, \infty[$ as $z \to \infty$. In this case, the dominated convergence theorem implies that

$$\begin{aligned} \lim_{\epsilon\to 0} \mathbb{P}_x(t < m | I^X < \epsilon) &= \lim_{\epsilon\to 0} \frac{1}{\mathbb{P}_x(I^X < \epsilon)}\, \mathbb{E}_x(g(X_t/\epsilon)) \\ &= \lim_{\epsilon\to 0} \left(\frac{\epsilon^{\gamma}}{\mathbb{P}_x(I^X < \epsilon)}\right) \mathbb{E}_x\left(X_t^{-\gamma} L(\epsilon/X_t)\right) \\ &= x^{\gamma}\, \mathbb{E}_x(X_t^{-\gamma}). \end{aligned}$$

However, in the case where Cramér's condition is not satisfied it follows from hypothesis (H') that $g$ is regularly varying at infinity with index $\gamma$ and we claim that $\mathbb{E}_x(X_s^{-\gamma-1}) < \infty$ for $x > 0, t \geq 0$, which, in view of Proposition 3 in [8], imply that

$$\lim_{\epsilon\to 0} \frac{1}{g(1/\epsilon)}\, \mathbb{E}_x(g(X_t/\epsilon)) = \mathbb{E}_x(X_t^{-\gamma}),$$

and the limit in equation (4.11) follows. So we just have to prove that $\mathbb{E}_x(X_s^{-\gamma-1}) < \infty$ for $x > 0, t \geq 0$. Indeed, we have seen that hypothesis (H') implies that $\mathbf{E}(e^{\gamma\xi_t}) \leq 1$, for all $t \geq 0$, and since Cramér's condition is not satisfied the latter inequality is a strictly one. So, by Lamperti's transformation

$$\begin{aligned}\int_0^\infty \mathrm{d}t\, \mathbb{E}_x(X_t^{-\gamma-1}) &= x^{-(\gamma+1)}\, \mathbf{E}\left(\int_0^\infty \mathrm{d}t \exp\{-(\gamma+1)\xi_{\tau(tx^{-1})}\}\right)\\ &= x^{-\gamma} \int_0^\infty \mathrm{d}s\, \mathbf{E}(e^{-\gamma\xi_s})\\ &= x^{-\gamma}(-\log(\mathbf{E}(e^{-\gamma\xi_1}))) < \infty, \qquad x > 0.\end{aligned}$$

Thus for $x > 0$, $\mathbb{E}_x(X_t^{-\gamma-1}) < \infty$, for a.e. $t > 0$, and by the scaling property the latter holds for any $t > 0, x > 0$.

To conclude, let $F_t \in \mathcal{F}_t$, $t > 0$, then arguing as before and using Fatou's lemma we have that

$$\begin{aligned}\liminf_{\epsilon\to 0} \mathbb{P}_x(F_t \cap \{t < m\} | I^X < \epsilon) &= \liminf_{\epsilon\to 0} \left(\frac{\mathbb{P}_1(I^X < \epsilon)}{\mathbb{P}_x(I^X < \epsilon)}\right) \mathbb{E}_x\left(1_{F_t} \frac{\mathbb{P}_{X_t}(I^X < \epsilon)}{\mathbb{P}_1(I^X < \epsilon)}\right)\\ &\geq x^\gamma\, \mathbb{E}_x(1_{F_t} X_t^{-\gamma}).\end{aligned}$$

Furthermore, applying this estimate to the set complementary of $F_t$ and using the result in equation (4.11) we get that

$$\limsup_{\epsilon\to 0} \mathbb{P}_x(F_t \cap \{t < m\} | I^X < \epsilon) \leq x^\gamma\, \mathbb{P}_x(F_t X_t^{-\gamma}).$$

□

It is interesting to note that in the non-Cramér case the law $\mathbb{P}^\downarrow$ is that of a pssMp that hits 0 in finite time and it does it by a jump,

$$\mathbb{P}_x^\downarrow(T_0 < \infty, X_{T_0-} > 0) = 1, \qquad \forall x > 0.$$

Roughly speaking, Theorem 5 tells us that in this case by pulling down the trajectory of $(X, \mathbb{P})$, under the law $\mathbb{P}$., from the place at which it attains its overall infimum for the last time, we break this trajectory and introduce a jump to the level 0.

However, the equality in (ii) Theorem 5 does not hold on the whole $\sigma$-field of the events prior to $m$, i.e. $\mathcal{F}_{m-} = \sigma\left(F_t \cap \{t < m\}, F_t \in \mathcal{F}_t, t \geq 0\right)$. Indeed, if this were the case it would imply that

$$\lim_{\epsilon\to 0} \mathbb{P}_x(X_{m-} \in \mathrm{d}y | I^X < \epsilon) = \mathbb{P}_x^\downarrow(X_{T_0-} \in \mathrm{d}y),$$

given that $X_{m-}$ is $\mathcal{F}_{m-}$–measurable. But the r.h.s. in the previous equality is equal to $\mathbf{P}^\downarrow(x\exp\{\xi_{\mathrm{e}}\} \in \mathrm{d}z)$, where e is a random variable independent of $\xi^\downarrow$ and with an exponential law of parameter $\kappa = -\log(e^{-\gamma\xi_1})$. While the l.h.s. is equal to the Dirac mass at 0 whenever 0 is regular for $(-\infty, 0)$.

**Remark 1.** Owing to the equivalent formulation of hypothesis (H') in terms of the underlying Lévy process it is easy to provide examples of pssMp that satisfies (H'). Indeed, it is easily deduced from Proposition 3 that when the process has no negative jumps the function $\varphi$ has the properties required in (H'). Besides, if a Lévy process does satisfies the hypotheses (a1) and Cramér's condition in (a2) of Theorem 5, it follows from the result of Bertoin and Doney [5] that

$$\lim_{t\to\infty} e^{\gamma t}\,\mathbf{P}(I^{\xi} < -t) = C,$$

where $C < \infty$ and $C > 0$ if and only if $\mathbf{E}(\xi_1^- e^{-\gamma\xi_1}) < \infty$. We deduce therefrom that under (a1) and (a2) of Theorem 5 we have

$$\mathbb{P}_x(I^X < \epsilon) \sim \epsilon^{\gamma} x^{-\gamma} C, \qquad \text{as } \epsilon \to 0,$$

and hence (H') is satisfied. Furthermore, the hypothesis (H') holds if the distribution of the negative of the overall minimum of $(\xi, \mathbf{P})$ belongs to a class of close to exponential laws $\mathcal{S}^{\gamma}$ with $\gamma > 0$. (See the recent work [21] for the definition of the classes $\mathcal{S}^{\gamma}$ and NASC on the Lévy process $(\xi, \mathbf{P})$ that ensure that the negative of the overall infimum belongs to one of this classes.)

# 5 Conditioning a pssMp to hit 0 continuously

Throughout this section we will assume that $(X, \mathbb{P})$ is a self–similar Markov process that belongs to the class (LC1). It was showed by Lamperti [22] that under these assumptions the process $(X, \mathbb{P})$ is the exponential of a Lévy process that has been killed at an independent exponential time and time changed, see Section 2 for more details. So, for notational convenience we will hereafter assume that $(\xi, \mathbf{P})$ is a Lévy process (with infinite lifetime), that $\boldsymbol{e}$, is an independent r.v. that follows an exponential law of rate $\mathrm{q} > 0$, and that the Lévy process with finite lifetime associated to $(X, \mathbb{P})$ via Lamperti's transformation is the one obtained by killing $(\xi, \mathbf{P})$ at time $\boldsymbol{e}$.

The problem of conditioning a self–similar Markov process that hits 0 by a jump to hit 0 continuously is a problem that was studied by Chaumont [9] in the case where the process has furthermore stationary and independent increments, i.e. is a stable Lévy process. See Chaumont and Caballero [12] for a computation of the underlying Lévy process of this pssMp in Lamperti's representation.

Throughout this section we will assume that

$$(\text{H"}) \;=\; \begin{cases} \text{non–arithmetic} \\ \text{there exists a } \gamma < 0 \text{ for which } \mathbf{E}(e^{\gamma\xi_1}) = e^{\mathrm{q}}, \\ \mathbf{E}(\xi_1^- e^{\gamma\xi_1}) < \infty. \end{cases}$$

Under these hypotheses we will prove the existence of a self–similar Markov process $(X, \mathbb{P}^{\downarrow})$ that can be thought as $(X, \mathbb{P})$ conditioned to hit 0 continuously.

The second hypothesis in (H") implies that the function $h^{\downarrow}(x) = e^{\gamma x}, x \in \mathbb{R}$ is an invariant function for the semigroup of $(\xi, \mathbf{P})$, killed at time $\boldsymbol{e}$. Let $\mathbf{P}^{\downarrow}$ be the $h$–transform of the law of $(\xi, \mathbf{P})$ killed at time $\boldsymbol{e}$, via the invariant function $h^{\downarrow}$. Under $\mathbf{P}^{\downarrow}$ the canonical process is still a

Lévy process with infinite lifetime that drifts to $-\infty$. Furthermore, by the third hypothesis in (H") we have that $m^{\downarrow} = \mathbf{E}^{\downarrow}(\xi_1) \in ]-\infty, 0[$. We are interested in the pssMp $(X, \mathbb{P}^{\downarrow})$, which is the Markov process associated to the Lévy process with law $\mathbf{P}^{\downarrow}$ via Lamperti's transformation. Since the Lévy process $(\xi, \mathbf{P}^{\downarrow})$ drifts to $-\infty$ we have that $(X, \mathbb{P}^{\downarrow}{}_x)$ hits 0 continuously at some finite time a.s. for every $x > 0$. As a consequence of the following result we will refer to $(X, \mathbb{P}^{\downarrow})$ as the process $(X, \mathbb{P})$ conditioned to hit 0 continuously.

**Theorem 6.** *Assume that the hypotheses (H") are satisfied.*

(*i*) *For every $x > 0$, $\mathbb{P}^{\downarrow}_x$ is the unique measure such that for every stopping time $T$ of $(\mathcal{G}_t)$ we have*

$$\mathbb{P}^{\downarrow}_x(F_T, T < T_0) = x^{-\gamma}\, \mathbb{P}_x(F_T X_T^{\gamma}, T < T_0),$$

*for every $F_T \in \mathcal{G}_T$.*

(*ii*) *For every $x > 0$,*

$$\lim_{\epsilon \to 0} \mathbb{P}_x(F_t \cap \{t < T_0\} | X_{T_0-} \leq \epsilon) = \mathbb{P}^{\downarrow}_x(F), \qquad F_t \in \mathcal{G}_t,\ t \geq 0.$$

(*iii*) *For every $x > 0$,*

$$\lim_{\epsilon \to 0} \mathbb{P}_x(F_t \cap \{t < T_0\} | \inf_{0 \leq t < T_0} X_t < \epsilon) = \mathbb{P}^{\downarrow}_x(F), \qquad F_t \in \mathcal{G}_t,\ t \geq 0.$$

*Proof.* Part (*i*) is an immediate consequence of the fact that $\mathbf{P}^{\downarrow}$ is an $h$–transform. To prove (ii) we will need the following Lemma in which we determine the tail distribution of a Lévy process at given exponential time.

**Lemma 5.** *Let $\sigma$ be a Lévy process of law $P$, and with infinite lifetime. Assume that $\sigma$ is non–arithmetic and that there exists a $\vartheta > 0$ for which $1 < E(e^{\vartheta \sigma_1}) < \infty$, and $E(\sigma_1^+ e^{\vartheta \sigma_1}) < \infty$. Let $T_\lambda$ be an exponential random variable of parameter $\lambda = \log E(e^{\vartheta \sigma_1})$ and independent of $\sigma$. We have that*

$$\lim_{x \to \infty} e^{\vartheta x} P(\sigma_{T_\lambda} \geq x) = \frac{\lambda}{\mu^{\natural} \vartheta},$$

*with $\mu^{\natural} = \mathbf{E}(\sigma_1 e^{\vartheta \sigma_1})$.*

Lemma 5 is a consequence of the renewal theorem for real–valued random variables and Cramer's method, see e.g. Feller [19] §XI.6.

*Proof.* Observe that the function $Z(x) = P(\sigma_{T_\lambda} \geq x)$, satisfies a renewal equation. More precisely, for $z(x) = \int_0^1 dt \lambda e^{-\lambda t} P(\sigma_t \geq x)$ and $L(dy) = e^{-\lambda} P(\sigma_1 \in dy)$ we have that

$$Z(x) = z(x) + \int_{-\infty}^{\infty} L(dy) Z(x - y).$$

This is an elementary consequence of the fact that the process $(\sigma'_s = \sigma_{1+s} - \sigma_1, s \geq 0)$ is a Lévy process independent of $(\sigma_r, r \leq 1)$ with the same law as $\sigma$. Next, the measure $L$ is a defective law, $L(\mathbb{R}) < 1$, such that

$$\int_{-\infty}^{\infty} e^{\vartheta y} L(dy) = e^{-\lambda} E(e^{\vartheta \sigma_1}) = 1; \quad \text{and} \quad \int_{-\infty}^{\infty} y e^{\vartheta y} L(dy) < \infty,$$

by hypotheses. Thus the function $Z^{\natural}(x) \equiv e^{\theta x}Z(x), x \in \mathbb{R}$ satisfies a renewal equation with $L(\mathrm{d}y)$ replaced by $L^{\natural}(\mathrm{d}y) = e^{\theta y}L(\mathrm{d}y)$, $y \in \mathbb{R}$, and $z$ replaced by $z^{\natural}(x) = e^{\theta x}z(x)$, $x \in \mathbb{R}$. By the uniqueness of the solution of the renewal equation we have that

$$Z^{\natural}(y) = \int_{\mathbb{R}} z^{\natural}(y-x)U^{\natural}(\mathrm{d}x), \qquad y \in \mathbb{R},$$

where $U^{\natural}(\mathrm{d}x)$ is the renewal measure associated to the law $L^{\natural}$. Furthermore, the function $z^{\natural}$ is directly Riemann integrable because it is the product of an exponential function and a decreasing one and $z^{\natural}$ is integrable. To see that $z^{\natural}$ is integrable, use the Fubini's theorem to establish

$$\begin{aligned}\int_{-\infty}^{\infty} z^{\natural}(x)dx &= \int_0^1 dt\lambda e^{-\lambda t}E\left(\int_{-\infty}^{\infty} dx e^{\vartheta x}1_{\{\sigma_t \geq x\}}\right) \\ &= \frac{1}{\vartheta}\int_0^1 dt\lambda e^{-\lambda t}E(e^{\vartheta\sigma_t}) \\ &= \frac{\lambda}{\vartheta} < \infty.\end{aligned}$$

Finally, given that $L^{\natural}$ is a non-defective law with finite mean the Key renewal theorem implies that

$$\lim_{y\to\infty} Z^{\natural}(y) = \lim_{y\to\infty}\int_{\mathbb{R}} z^{\natural}(y-x)U^{\natural}(\mathrm{d}x) = \frac{1}{\mu^{\natural}}\int_{-\infty}^{\infty} z^{\natural}(x)dx = \frac{\lambda}{\vartheta\mu^{\natural}}.$$

□

Now we may end the proof of part $(ii)$. Observe that under $\mathbb{P}_x$ the random variable $X_{T_0-}$ has the same law as $xe^{\xi_{\boldsymbol{e}}}$ under $\mathbf{P}$. Then, applying Lemma 5 to $(-\xi, \mathbf{P})$ we obtain by hypotheses (H") that

$$\lim_{y\to\infty} e^{-\gamma y}\,\mathbf{P}(\xi_{\boldsymbol{e}} \leq -y) = \frac{\mathrm{q}}{\gamma\mu^{\downarrow}} := \mathrm{d}_{\mathrm{q}},$$

with $\mu^{\downarrow} = \mathbf{E}(\xi_1 e^{\gamma\xi_1}) \in ]-\infty, 0[$, which is finite by hypothesis. Thus, we have the following estimate of the left tail distribution of $X_{T_0-}$

$$\lim_{\epsilon\to 0}\epsilon^{\gamma}\,\mathbb{P}_x(X_{T_0-} \leq \epsilon) = x^{\gamma}\mathrm{d}_{\mathrm{q}}. \tag{5.1}$$

We conclude by a standard application of the Markov property, estimate (5.1) and a dominated convergence argument.

Now we prove part $(iii)$. First of all, we claim that under the assumptions of Theorem 6,

$$x^{-\gamma}\lim_{\epsilon\to 0+}\epsilon^{\gamma}\,\mathbb{P}_x(\inf_{0\leq t<T_0} X_t < \epsilon) := d^{"}_{\mathrm{q}} \in ]0,\infty[, \qquad x > 0. \tag{5.2}$$

Owing to this estimate the rest of the proof of Theorem 6 (iii) is quite similar to the one of (ii) in Theorem 5 in the case where Cramer's condition is satisfied, so we omit the details. Indeed, it is clear that the r.v. $\inf_{0\leq t<T_0} X_t$, has the same law as

$$\exp\{\inf_{0\leq s\leq \boldsymbol{e}}\{\xi_s\}\},$$

under $\mathbf{P}$. Its well known that $(\sup_{0\leq s\leq \mathbf{e}}\{-\xi_s\}, \mathbf{P})$ has the same law as a subordinator, say $\widetilde{\sigma}$, with Laplace exponent $\widehat{\kappa}(\mathrm{q},\lambda)-\widehat{\kappa}(\mathrm{q},0)$, evaluated at an independent exponential time of parameter $\widehat{\kappa}(\mathrm{q},0)$, where $\widehat{\kappa}(\cdot,\cdot)$ is the bivariate Laplace exponent of the dual ladder height process associated to $(\xi,\mathbf{P})$, see e.g. [3] Section VI.2. So in order to deduce the assertion (5.2) using Lemma 5 we have to verify that

$$\text{(a) } 1<\mathbf{E}(e^{\widehat{\gamma}\widetilde{\sigma}_1})<\infty, \quad \text{(b) } \mathbf{E}(\widetilde{\sigma}_1 e^{\widehat{\gamma}\widetilde{\sigma}_1})<\infty \quad \text{and} \quad \text{(c) } \widehat{\kappa}(\mathrm{q},0)=\log \mathbf{E}(e^{\widehat{\gamma}\widetilde{\sigma}_1}), \quad \text{for } \widehat{\gamma}=-\gamma.$$

Recall that a function $f:\mathbb{R}\to\mathbb{R}$, of the type $f(x)=|x|^a e^{\beta x}$, for $a\in\mathbb{R}$, $\beta<0$, is integrable w.r.t. the law of $(\xi_t,\mathbf{P})$ for some $t>0$ if and only if $f(x)1_{\{x<-1\}}$ is integrable w.r.t. the Lévy measure of $(\xi,\mathbf{P})$, see e.g. [27] Proposition 25.4. Furthermore, Vigon [29] Section 6.2, established that $f(x)1_{\{x<-1\}}$, is integrable w.r.t. the Lévy measure of $(\xi,\mathbf{P})$ if and only if $f(-x)1_{\{-x>1\}}$, is integrable w.r.t. the Lévy measure of the dual ladder height subordinator associated to $(\xi,\mathbf{P})$. So, that (a) and (b) are consequences of the hypotheses (H") and the fact that the subordinator $\widetilde{\sigma}$ has the same Lévy measure and drift term as the dual ladder height subordinator associated to $(\xi,\mathbf{P})$. Finally, the assertion in (c) is an easy consequence of the inversion theorem in Vigon [29] Section 4.3. $\square$

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